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COMMENT

The asymptotic behaviour of directed self-avoiding walks

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Abstract. Chakrabarti and Manna have made a numerical study of 'directed' self-avoiding walks on a square lattice, in which the random walker is forbidden to move upward: extrapolating from an enumeration of N-step walks up to $N = 14$, they claimed that the mean end-to-end displacement, $\overline{R_N}$, behaves asymptotically for large N as N^{ν} with $\nu = 0.86 \pm 0.02$. Here it is shown that a simple application of the generating function method yields the behaviour of N-step walks of this type exactly (for all N), and the result $\nu = 1$ is rigorously proved.

In a recent letter to this journal, Chakrabarti and Manna (1983) described numerical studies of a 'new' class of 'directed' self-avoiding walks (SAWs) on a square lattice, in which the vertical steps of the random walker were permitted to be in the negative y direction only. Thus, a general walk of this type consists of any number, including zero, of horizontal steps to the left or to the right of the starting point, then a step downward, then again any number of steps to the left or to the right, then another step downward, and so on. By enumerating all such walks of N steps through $N = 14$ and extrapolating the results, Chakrabarti and Manna claimed that the mean end-to-end displacement, $\overline{R_N}$, of N-step walks behaves asymptotically as $R_N \sim N^{\nu}$ with $\nu = 0.86 \pm 0.02$. If true, this would place such walks in a new universality class, distinct from free two-dimensional walks (for which $\nu = \frac{1}{2}$) and ordinary two-dimensional SAWs (for which $\nu = \frac{3}{4}$ (Nienhuis 1982)).

In fact, however, this type of directed SAW was investigated some time ago by Temperley (1956) and by Fisher and Sykes (1959) as an approximate description of the boundary between two oppositely magnetised domains of an Ising model on a square lattice (whence Temperley called such walks 'Onsager boundaries'). A simple argument (given below) shows that for these directed SAWs one must have $\nu = 1$, as for an ordinary one-dimensional walk which proceeds only in one direction. The method of generating functions, discussed by Temperley (1956), allows one easily to convert this argument to a rigorous proof, and in fact also enables one to calculate exactly the mean-*square* end-to-end displacement, $\overline{R_N^2}$ (which is the more natural quantity to consider).

The fact that $\nu = 1$ follows from the observation that the constraint on vertical steps effectively eliminates all correlation between the horizontal and vertical motion of the walker. Thus, in the sum

$$\overline{R_N^2} = \overline{[R_N^{(h)}]^2} + \overline{[R_N^{(v)}]^2} \tag{1}$$

(where the superscripts h, v refer to the net horizontal and vertical displacements, respectively), one expects to find

$$\overline{[R_N^{(h)}]^2} \sim N^{2\nu_h} \qquad \text{with } \nu_h = \tfrac{1}{2}, \tag{2}$$

as for free walks on the line (which have, asymptotically, a Gaussian distribution), while

$$\overline{[R_N^{(v)}]^2} \sim N^{2\nu_v} \qquad \text{with } \nu_v = 1, \tag{3}$$

as for walks on the line which proceed in one direction only. As a result, $\overline{R_N^2} \sim N^{2\nu}$ (or $\overline{R_N} \sim N^{\nu}$) with $\nu = 1$.

To make this argument rigorous, assign a weight z to every vertical (downward) step, u to every left step, and v to every right step. Let $G^{(N)}_{n_l n_r}$ denote the number of N-step walks with n_l left and n_r right steps, and define the generating function

$$G(z, u, v) = \sum_{N, n_l, n_r} G^{(N)}_{n_l n_r} z^{N-n_l-n_r} u^{n_l} v^{n_r}. \tag{4}$$

By considering the construction of a general walk as described in the first paragraph, it is clear that (Temperley 1956)

$$G(z, u, v) = H[1 + zH + (zH)^2 + (zH)^3 + \ldots] = H(1 - zH)^{-1} \tag{5}$$

where

$$H(u, v) = 1 + (u + u^2 + u^3 + \ldots) + (v + v^2 + v^3 + \ldots) = (1 - uv)(1-u)^{-1}(1-v)^{-1}. \tag{6}$$

The total number G_N of N-step walks is then obtained from

$$G(z, z, z) \equiv \sum_{N=0}^{\infty} G_N z^N = (1+z)(1 - 2z - z^2)^{-1} \tag{7}$$

whence, by expanding the right-hand side about $z = 0$ and equating like powers of z, one obtains

$$\begin{aligned} G_N &= \tfrac{1}{2}[(1+\surd 2)^{N+1} + (1 - \surd 2)^{N+1}] && \text{for all } N \geq 0 \\ &\sim (1+\surd 2)^N && \text{as } N \to \infty. \end{aligned} \tag{8}$$

Now one establishes bounds for $\overline{R_N}$ by noting that

$$\overline{R_N^{(v)}} \leq \overline{R_N} \leq N, \tag{9}$$

where

$$\overline{R_N^{(v)}} \equiv \frac{1}{G_N} \sum_{n_l, n_r} G^{(N)}_{n_l n_r} (N - n_l - n_r). \tag{10}$$

Hence, from (4), one has

$$\sum_{N=0}^{\infty} G_N \overline{R_N^{(v)}} z^N = z \frac{\partial G}{\partial z}\bigg|_{z=u=v} = z(1+z)^2(1 - 2z - z^2)^{-2}, \tag{11}$$

so that, expanding (11) and using (8), one finds

$$\overline{R_N^{(v)}} = \frac{N-1}{2} + \frac{1}{2\surd 2}\left(\frac{(1+\surd 2)^{N+1} - (1-\surd 2)^{N+1}}{(1+\surd 2)^{N+1} + (1-\surd 2)^{N+1}}\right) = \tfrac{1}{2}N + \mathrm{O}(1). \tag{12}$$

Thus, recalling (9), the fact that $\nu = 1$ is proved.

It is clear that the mean-square displacement $\overline{R_N^2}$ can also be calculated exactly from appropriate (second) derivatives of G, analogous to the calculation of $\overline{R_N^{(v)}}$

above. The exact result is algebraically simple but rather long, so that only its asymptotic behaviour will be given here:

$$\overline{R_N^2} = \tfrac{1}{4}N^2 + \tfrac{7}{8}\sqrt{2}N + O(1). \tag{13}$$

The separate asymptotic contributions of $\overline{[R_N^{(v)}]^2} \sim N^2$ and $\overline{[R_N^{(h)}]^2} \sim N$, as noted earlier, are apparent in this result. Evidently, Chakrabarti and Manna were misled by examining a plot of $\log \overline{R_N}$ against $\log N$ over too small a range of values of N, for which the crossover from the N to the N^2 term gave the appearance of $R_N \sim N^{\nu'}$ with $\frac{1}{2} < \nu' < 1$. This error simply underscores the general rule that power-law behaviour cannot reliably be extracted from log–log plots of data unless the parameters vary over at least two or three decades.

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Note added in proof. This proof extends straightforwardly to general dimensions $d \geqslant 2$, where it continues to yield $G_N \sim N^{\gamma-1}\mu^N$ with $\gamma = 1$, as well as $\overline{[R_N^{(\parallel)}]^2} \sim N^{2\nu_\parallel}$ with $\nu_\parallel = 1$ and $\overline{[R_N^{(\perp)}]^2} \sim N^{2\nu_\perp}$ with $\nu_\perp = \frac{1}{2}$ (for the mean-square displacements respectively parallel and transverse to the restricted direction), regardless of the detailed self-avoiding nature of the walks in the transverse directions.

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